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Analysis of a General System for the Detection of Amplitude Modulated Noise

by

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ABSTRACT

A general system for the detection of amplitude-modulated noise, in the presence of background noise, is analyzed with a view toward determining the behavior and optimum design of the system. The unmodulated noise carrier and the background noise are assumed to be independent stationary Gaussian random time functions. The modulating function is a time function which, in general, is random, non-Gaussian, and nonstationary. A detection criterion, as a measure of the performance of the system, is defined, and computed in terms of the input power spectra and the transfer characteristics of the system. The techniques used, and the intermediate mathematical results obtained, are of interest in themselves. The results are applied to give a detailed analysis of a typical detection system which is a special case of the general system.

1. INTRODUCTION

The problem of the detection of signals having the character of amplitude modulated random noise is of recent interest in communications theory. For example, a paper by Deutsch[1] treats the effect of a linear detector on such signals when the modulation is periodic.

In this report, we treat a general detection system involving square-law detectors. We shall compute various statistics of the output, with and without modulation, and, from these, compute a detection criterion which we define as a measure of the detectability of the modulated noise.

The system is diagramed in Fig. 1. The input filters H_1 and H_2 include the characteristics of any receiving equipment which might ordinarily precede them. The only assumptions made on the filters K_1 and K_2 are that they do not pass dc. The output integrator is perfect with integration time T.

We take the input u(t) to be a stationary Gaussian noise y(t) modulated with index m by a signal function g(t), statistically independent of y(t). Thus

$$u(t) = y(t)\left[1 + mg(t)\right]. \tag{1.1}$$

We assume both the noise and the signal to have zero mean level; that is*,

$$\langle y(t)\rangle = 0, \quad \langle g(t)\rangle_{TAv} = 0. \tag{1.2}$$

Then, the total input power, P_u, is

$$P_u = \langle u^2(t)\rangle_{TAv} = P_n \quad (1 + m^2 P_g) \tag{1.3}$$

where

$$P_n = \langle y^2(t)\rangle = \int_0^\infty G_y(\omega)\, d\omega \tag{1.4}$$

$$P_g = \langle g^2(t)\rangle_{TAv} = \int_0^\infty G_g(\omega)\, d\omega$$

* This notation is defined in the Appendix in which we summarize the main notations and notions on random time functions that we use.

Thus $m^2 P_g$ is the ratio of the sideband power to the carrier power. We shall assume

$$m^2 P_g \ll 1, \qquad (1.5)$$

since this is the interesting case in a detection problem. The methods used, however, are applicable for any value of the ratio.

If by detection is meant the extraction of information about the modulating function (which would also seem to be the best means for distinguishing this type of signal from unmodulated noise), then, the inherent presence of fluctuation noise at the output of the detector, due to demodulation of the input noise carrier, makes this a problem of the detection of signals in the presence of additive noise.

In order to see this more clearly, and to better understand the action of the general system of Fig. 1, we shall compute the output power spectrum of a square-law detector when the input is unmodulated noise. This calculation also gives a simple illustration of the methods used in treating the general system.

2. POWER SPECTRUM AFTER A SQUARE-LAW DETECTOR

If the input of a square-law detector is given by (1.1), then, the output $v(t)$ of the squarer is given by

$$v(t) = u^2(t) = y^2(t)\left[1 + 2mg(t) + m^2g^2(t)\right]. \tag{2.1}$$

In order to compute the power spectrum $G_v(\omega)$ of $v(t)$, we first compute the autocorrelation $R_v(\tau)$, and then obtain $G_v(\omega)$ by means of the Wiener-Khintchine theorem (Eq. (A5) in the Appendix).

$$\begin{aligned} R_v(\tau) &= \langle v(t)\, v(t+\tau)\rangle_{TAv} \qquad (2.2)\\ &= \langle y^2(t)\, y^2(t+\tau)\rangle \Big\{1 + m^2\Big[\langle g^2(t)\rangle_{TAv} \\ &\quad + \langle g^2(t+\tau)\rangle_{TAv} + 4\langle g(t)\, g(t+\tau)\rangle_{TAv}\Big] \\ &\quad + m^4\langle g^2(t)\, g^2(t+\tau)\rangle_{TAv}\Big\}. \end{aligned}$$

The terms in (2.2) which involve m and m^3 vanish, since we assume that all time-ensemble averages of odd powers of g(t) vanish.

To compute the foregoing fourth order ensemble average of y(t), we use the formula (A7) given in the Appendix, which expresses the higher order moments of a Gaussian random time function in terms of its autocorrelation function $R_y(\tau)$.
We have

$$\begin{aligned} \langle y^2(t)\, y^2(t+\tau)\rangle &= \langle y^2(t)\rangle\langle y^2(t+\tau)\rangle + 2\{\langle y(t)\, y(t+\tau)\rangle\}^2 \qquad (2.3)\\ &= R_y^2(0) + 2R_y^2(\tau). \end{aligned}$$

As regards the fourth order time-ensemble average of g(t), we make the assumption (which is fulfilled by a large number of random time functions) that

$$\langle g^2(t)\, g^2(t+\tau)\rangle_{TAv} = R_g^2(0) + f(\tau) \tag{2.4}$$

where $f(\tau)$ is of the order no greater than that of $R_g^2(0)$.

For example, if g(t) has Gaussian statistics, then

$$f(\tau) = 2\, R_g^{\,2}(\tau) \leq 2\, R_g^{\,2}(0). \tag{2.5}$$

Consequently, we obtain from (2.2) that

$$R_v(\tau) = \left\{R_y^{\,2}(0) + 2\, R_y^{\,2}(\tau)\right\}\left\{\left[1 + m^2 R_g(0)\right]^2 + 4\, m^2 R_g(\tau) + m^4 f(\tau)\right\}. \tag{2.6}$$

In view of assumption (1.5), and the foregoing remarks about $f(\tau)$, we may henceforth ignore the term in (2.6) involving m^4.

The power spectrum $G_v(\omega)$ is now easily obtained by taking the Fourier integral of (2.6). The resulting terms group themselves naturally into three groups; by (1.4), we write P_n and P_g for $R_y(0)$ and $R_g(0)$ respectively.

The power spectrum $G_v(\omega)$ is the sum of dc terms: (2.7)

$$2\,\delta(\omega)\, P_n^{\,2}\left[1 + m^2 P_g\right]^2 ,$$

steady state or signal terms: $4\, m^2 P_n^{\,2} G_g(\omega)$,

noise or fluctuation terms: $\left[1 + m^2 P_g\right]^2 \int d\mu\, G_y(\mu)\, G_y(\omega - \mu)$

$$+ 2\, m^2 \int d\nu\, G_g(\nu) \int d\mu\, G_y(\mu)\, G_y(\mu + \omega - \nu).$$

The integrated power is then, dropping terms involving $m^2 P_g$ by assumption (1.5),

dc	$P_n^{\,2}$	(2.8)
signal	$4\, m^2 P_n^{\,2} P_g$	
noise	$2\, P_n^{\,2}$.	

Thus, as stated at the end of Sec. 1, at the output of the square-law detector the problem reduces to that of detecting a signal of total power $4\, m^2\, P_n\, P_g$ in the presence of an additive noise of total power $2\, P_n^2$.

This problem is usually attacked by analyzing the detector output with a band-pass or low-pass filter followed by squaring and averaging to determine the mean square signal and noise.

It is easily seen that the complete detection system, diagramed in Fig. 2, consisting of receiver, square-law detector, filter, squarer, and finite time integrator is a special case of the system of Fig. 1, obtained by setting $H = H_1 = H_2$ and $K = K_1 = K_2$. This case is treated in detail in Sec. 8.

3. THE DETECTION CRITERION

Suppose that it is possible to measure the output $\overline{w}(T)$ of the system of Fig. 1 when the input is unmodulated noise. A large number of such measurements will have an average and a mean square fluctuation given by the ensemble average $\langle \overline{w}(T) \rangle_n$ and the variance

$$\sigma_n^2 = \langle \overline{w}^2(T) \rangle_n - \langle \overline{w}(T) \rangle_n^2 \tag{3.1}$$

Similarly, a large number of measurements of the output when the input is amplitude modulated noise will have mean value $\langle \overline{w}(T) \rangle_{mn}$ and variance

$$\sigma_{mn}^2 = \langle \overline{w}^2(T) \rangle_{mn} - \langle \overline{w}(T) \rangle_{mn}^2 \quad . \tag{3.2}$$

The subscript n on a quantity is to indicate that the quantity is to be evaluated under the assumption that the input to the system is unmodulated noise, while the subscript mn will be used to indicate that the quantity is to be evaulated under the assumption that the input is amplitude modulated noise.

Under the assumption that the observations are independent, the difference $\Delta(T)$ of the observed outputs in the two cases,

$$\Delta(T) = \overline{w}(T)_{mn} - \overline{w}(T)_n \ , \tag{3.3}$$

will have mean value $\langle \Delta(T) \rangle = \langle \overline{w}(T) \rangle_{mn} - \langle \overline{w}(T) \rangle_n$ and variance equal to $\sigma_{mn}^2 + \sigma_n^2$. Using the statistical theory of testing hypotheses, if the probability distribution of $\Delta(T)$ is known, by testing the hypothesis that $\langle \Delta(T) \rangle = 0$ one can detect the presence or the absence of signal. To any preassigned probability p one can find a number, K, such that

$$|\Delta(T)| \leq K \left[\sigma_{mn}^2 + \sigma_n^2\right]^{1/2} \tag{3.4}$$

with probability p. As the detection process, one then adopts the

following rule. One decides that noise alone is present if the difference of the observed outputs lies in the range given by Eq. (3.4), and that signal (in our case, modulated noise) is present if the difference of the observed outputs lies outside the range given by Eq. (3.4).

Motivated by the foregoing considerations, we therefore adopt as our detection criterion, denoted by D(T), the following ratio:

$$D(T) = \frac{\langle \bar{w}(T) \rangle_{mn} - \langle \bar{w}(T) \rangle_{n}}{\left[\sigma_{mn}^{2} + \sigma_{n}^{2} \right]^{1/2}} \tag{3.5}$$

It is presumed that the larger D(T) is for a given detection system, the better will be the system's performance, and the more likely it will be to detect the presence of signal.

In many cases, it can be assumed that the fluctuation term for the case of signal and noise in the input is roughly equal to the fluctuation term σ_n for the case of noise alone in the input. We then have for the detection criterion

$$D(T) = \frac{\langle \bar{w}(T) \rangle_{mn} - \langle \bar{w}(T) \rangle_{n}}{\sqrt{2}\ \sigma_{n}} \tag{3.6}$$

In view of assumption (1.5), one could show that σ_{mn} is roughly equal to σ_n, and we therefore use Eq. (3.6) for the detection criterion.

We now turn our attention to the problem of expressing D(T) in terms of the statistics of the noise and of the modulating function. This problem is solved in several stages, as follows.

In Sec. 4, we express D(T) for large T in terms of the cross-correlation, $\rho(\tau)$, of the outputs, $\bar{v}_1(t)$ and $\bar{v}_2(t)$, after the second filters, and the autocorrelation, $R_w(\tau)$, of the output, w(t), of the multiplier. We also obtain a similar, but more complicated, expression for D(T) for small T.

In Sec. 5, the cross-correlation, $\rho(\tau)$, is computed under two different assumptions. First, it is computed under the assumption that the input of the system is unmodulated stationary Gaussian noise, in which case we denote it by $\rho_n(\tau)$. Since the stationary character of Gaussian noise is preserved after passage through non-linear devices, it holds that

$$\rho_n(\tau) = \left[\langle \bar{v}_1(t)\ \bar{v}_2(t+\tau)\rangle_{TAv}\right]_n \tag{3.7}$$

$$= \langle \bar{v}_1(t)\ \bar{v}_2(t+\tau)\rangle_n$$

Second, $\rho(\tau)$ is computed under the assumption that the input, u(t), is amplitude-modulated noise. In this case, we denote it by $\rho_{mn}(\tau)$.

Thus,

$$\rho_{mn}(\tau) = \left[\langle \bar{v}_1(t)\ \bar{v}_2(t+\tau)\rangle_{TAv}\right]_{mn} . \tag{3.8}$$

In Sec. 6, the autocorrelation, $R_w(\tau)$, is computed under the assumption that the input is noise. We have therefore

$$R_w(\tau) = \left[\langle w(t)\ w(t+\tau)\rangle_{TAv}\right]_n \tag{3.9}$$

$$= \langle w(t)\ w(t+\tau)\rangle_n$$

Finally, in Sec. 7, the results of the preceding sections are combined to yield the required expression for D(T).

4. LIMITING FORMS FOR THE DETECTION CRITERION

A Formula for $D^2(T)$

Since

$$\bar{w}(T) = \int_0^T w(t)dt = \int_0^T \bar{v}_1(t)\,\bar{v}_2(t)dt, \tag{4.1}$$

it follows that

$$\langle \bar{w}(T) \rangle_n = \int_0^T \langle \bar{v}_1(t)\,\bar{v}_2(t) \rangle_n \, dt = T \rho_n(0), \tag{4.2}$$

$$\langle \bar{w}(T) \rangle_{mn} = \int_0^T \langle \bar{v}_1(t)\,\bar{v}_2(t) \rangle_{mn} \, dt, \tag{4.3}$$

$$\langle \bar{w}^2(T) \rangle_n = \int_0^T \int_0^T dt\, dt' \, \langle w(t)\, w(t') \rangle_n \tag{4.4}$$

$$= \int_0^T \int_0^T dt\, dt' \, R_w(t'-t) = 2\int_0^T d\tau\, R_w(\tau)\,(T-\tau).$$

Therefore,

$$\langle \bar{w}^2(T) \rangle_n - \langle \bar{w}(T) \rangle_n^2 = 2\int_0^T d\tau\, R_w(\tau)\,(T-\tau) \tag{4.5}$$

$$- \left[T \rho_n(0)\right]^2 = 2\int_0^T d\tau \left[R_w(\tau) - \rho_n^2(0)\right](T-\tau).$$

Consequently we may write for $D^2(T)$:

$$\frac{2\,D^2(T)}{T} = \frac{\left\{\frac{1}{T}\int_0^T \langle \bar{v}_1(t)\,\bar{v}_2(t) \rangle_{mn} dt - \rho_n(0)\right\}^2}{2\int_0^T \left(1 - \tau/T\right)\left[R_w(\tau) - \rho_n^2(0)\right] d\tau}. \tag{4.6}$$

The Limit for T Large

If the noise, y(t), has a continuous power spectrum $G_y(\omega)$, as we assume to be the case, then it will be shown in Sec. 6 that the power spectrum $G_W(\omega)$ of the output of the multiplier (when the input is noise) will also be continuous except for a dc delta-function term equal to $2\rho_n^2(0)$. We therefore define

$$G'_w(\omega) = G_w(\omega) - 2\rho_n^2(0)\,\delta(\omega) \tag{4.7}$$

to denote the continuous power spectrum of the output w(t) of the multiplier.

The autocorrelation, $R'_W(\tau)$, corresponding to $G'_W(\omega)$ is then given by

$$R'_W(\tau) = R_W(\tau) - \rho_n^2(0). \tag{4.8}$$

$R'_W(\tau)$, unlike $R_W(\tau)$, has a finite integral from 0 to ∞. By the Wiener-Khintchine theorem, it follows that

$$G'_w(0) = \frac{2}{\pi}\cdot\int_0^\infty R'_w(\tau)\,d\tau = \frac{2}{\pi}\int_0^\infty\left[R_w(\tau) - \rho_n^2(0)\right]d\tau \tag{4.9}$$

Let us now pass to the limit, as $T \to \infty$, in (4.6). By definition, the first term of the numerator tends to $\rho_{mn}(0)$. From the fact that $R'_W(\tau)$ is integrable it follows that the denominator tends to $\pi G'_W(0)$.

We therefore obtain that

$$\lim_{T\to\infty} 2\,\frac{D^2(T)}{T} = \frac{\left[\rho_{mn}(0) - \rho_n(0)\right]^2}{\pi\, G'_w(0)} \tag{4.10}$$

Thus for T large, we have approximately

$$\mathcal{D}(T) = \sqrt{T} \quad \frac{\rho_{mn}(0) - \rho_n(0)}{[2\pi G'_w(0)]^{1/2}} \,. \tag{4.11}$$

Thus the detection ratio increases with the square root of the integration time. In other words, in order to double the detectability of a weak signal in the presence of noise, it is necessary to quadruple the integration time.

The Limit for T Small

For T small, the denominator of Eq. (4.6) is approximately $T\left[R_w(0) - \rho_n^{\,2}(0)\right]$. However, there is no simple expression for the first term of the numerator, unless the signal is stationary, when the integration is no longer necessary. We therefore can only obtain the formal expression

$$\lim_{T\to 0} \; 2\,\mathcal{D}^2(T) = \frac{\left\{ \lim\limits_{T\to 0} \frac{1}{T} \int_0^T \langle \bar{v}_1(t)\, \bar{v}_2(t)\rangle_{mn}\, dt - \rho_n(0) \right\}^2}{R_w(0) - \rho_n^{\,2}(0)} \tag{4.12}$$

In Sec. 5, we will show how the first term in the numerator of Eq. (4.12) may be evaluated if we possess a knowledge of the complete statistics of g(t); that is, if we know the mean value function

$$\mu_g(t) = \langle g(t) \rangle \tag{4.13}$$

and the covariance function

$$\Gamma_g(t_1, t_2) = \langle g(t_1) \;\; g(t_2) \rangle \,. \tag{4.14}$$

It may be noted that the assumptions we have already made on g(t) may be written

$$0 = \left[\mu_g(t)\right]_{TAv} \tag{4.15}$$

$$R_g(\tau) = \left[\Gamma_g(t, t+\tau)\right]_{TAv} \tag{4.16}$$

5. CROSS-CORRELATION BEFORE THE MULTIPLIER

In this section, we compute $\rho_n(\tau)$ and $\rho_{mn}(\tau)$. We make the convention that all integrals which are taken from $-\infty$ to ∞ are to be written without the limits of integration. Since

$$\bar{v}_1(t) = \int d\xi\, k_1(\xi) \iint d\alpha_1\, d\alpha_2\, h_1(\alpha_1)\, h_1(\alpha_2)\, u(t-\alpha_1-\xi)\, u(t-\alpha_2-\xi) \quad (5.1)$$

$$\bar{v}_2(t+\tau) = \int d\eta\, k_2(\eta) \iint d\beta_1\, d\beta_2\, h_2(\beta_1)\, h_2(\beta_2)\, u(t+\tau-\beta_1-\eta)\, u(t+\tau-\beta_2-\eta) \quad (5.2)$$

it follows that

$$\rho_n(\tau) = \int' \langle F_1[y(t)] \rangle \quad (5.3)$$

$$\rho_{mn}(\tau) = \int' \langle F_1[y(t)] \rangle \langle F_1[1+mg(t)] \rangle_{TAv} \quad (5.4)$$

where for brevity we use the single primed integral sign $\int'$ to denote the six fold integration,

$$\iint d\xi\, d\eta\, k_1(\xi)\, k_2(\eta) \iiiint d\alpha_1\, d\alpha_2\, d\beta_1\, d\beta_2\, h_1(\alpha_1)\, h_1(\alpha_2)\, h_2(\beta_1)\, h_2(\beta_2) \quad (5.5)$$

and for a function f(t), we define the notation $F_1[f(t)]$ by

$$F_1[f(t)] = f(t-\alpha_1-\xi)\, f(t-\alpha_2-\xi)\, f(t+\tau-\beta_1-\eta)\, f(t+\tau-\beta_2-\eta) \quad (5.6)$$

In Eq. (5.4), we have used the formula

$$\langle F_1[y(t)]\, F_1[1+mg(t)] \rangle_{TAv} = \langle F_1[y(t)] \rangle \langle F_1[1+mg(t)] \rangle_{TAv} \quad (5.7)$$

which follows from the statistical independence of y(t) and g(t), and the stationary nature of y(t).

We henceforth drop the subscript y in writing the autocorrelation and power spectrum of y(t). Since y(t) is Gaussian, we may express the four fold ensemble average $\langle F_1 [y(t)] \rangle$ in terms of the autocorrelation $R(\tau)$ of y(t) by means of Eq. (A7) given in the Appendix. We thus obtain

$$\begin{aligned} \langle F_1[y(t)] \rangle = \; & R(\alpha_1-\alpha_2)\, R(\beta_1-\beta_2) \\ & + R(\tau+\alpha_1+\xi-\beta_1-\eta)\, R(\tau+\alpha_2+\xi-\beta_2-\eta) \\ & + R(\tau+\alpha_1+\xi-\beta_2-\eta)\, R(\tau+\alpha_2+\xi-\beta_1-\eta). \end{aligned} \tag{5.8}$$

$F_1 \left[1+mg(t) \right]$ is easily expanded in powers of m, and we obtain

$$\begin{aligned} F_1[1+mg(t)] = \; & 1 \\ & +m\left\{ g(t-\alpha_1-\xi) + g(t-\alpha_2-\xi) + g(t+\tau-\beta_1-\eta) + g(t+\tau-\beta_2-\eta) \right\} \\ & +m^2\left\{ g(t-\alpha_1-\xi)\, g(t-\alpha_2-\xi) + g(t-\alpha_1-\xi)\, g(t+\tau-\beta_1-\eta) \right. \\ & \quad + g(t-\alpha_1-\xi)\, g(t+\tau-\beta_2-\eta) + g(t-\alpha_2-\xi)\, g(t+\tau-\beta_1-\eta) \\ & \quad \left. + g(t-\alpha_2-\xi)\, g(t+\tau-\beta_2-\eta) + g(t+\tau-\beta_1-\eta)\, g(t+\tau-\beta_2-\eta) \right\} \\ & +m^3\left\{ g(t-\alpha_1-\xi)\, g(t-\alpha_2-\xi)\, g(t+\tau-\beta_1-\eta) \right. \\ & \quad + g(t-\alpha_1-\xi)\, g(t-\alpha_2-\xi)\, g(t+\tau-\beta_2-\eta) \\ & \quad + g(t-\alpha_1-\xi)\, g(t+\tau-\beta_1-\eta)\, g(t+\tau-\beta_2-\eta) \\ & \quad \left. + g(t-\alpha_2-\xi)\, g(t+\tau-\beta_1-\eta)\, g(t+\tau-\beta_2-\eta) \right\} \\ & +m^4\left\{ g(t-\alpha_1-\xi)\, g(t-\alpha_2-\xi)\, g(t+\tau-\beta_1-\eta)\, g(t+\tau-\beta_2-\eta) \right\}. \end{aligned} \tag{5.9}$$

Upon taking the time-ensemble average, we obtain

$$\langle F_1[1+mg(t)]\rangle_{TAv} = 1 \tag{5.10}$$
$$+ m^2\{R_g(\alpha_1-\alpha_2) + R_g(\beta_1-\beta_2) + R_g(\tau+\alpha_1+\xi-\beta_1-\eta)$$
$$+ R_g(\tau+\alpha_1+\xi-\beta_2-\eta) + R_g(\tau+\alpha_2+\xi-\beta_1-\eta) + R_g(\tau+\alpha_2+\xi-\beta_2-\eta)\}$$
$$+ m^4\langle g(t-\alpha_1-\xi)\,g(t-\alpha_2-\xi)\,g(t+\tau-\beta_1-\eta)\,g(t+\tau-\beta_2-\eta)\rangle_{TAv}$$

The terms in m and m^3 vanish, because it has been assumed that all time-ensemble averages of odd powers of g(t) vanish. In what follows, we may also ignore the term in Eq. (5.10) involving m^4, since we assume that the fourth order time-ensemble average of g(t) is of the order of $R_g^{\,2}(0)$. Consequently, the term in Eq. (5.8) involving m^4 is of the order of $\left[m^2R_g(0)\right]^2$, and this term may be dropped in view of the discussion in Sec. 1.

In view of Eqs. (5.10), (5.8), (5.4), and (5.3), we could now write an expression for $\rho_n(\tau)$ and for $\rho_{mn}(\tau)$ in terms of the modulating index, m, the autocorrelation functions, $R(\tau)$ and $R_g(\tau)$, and the filter impulse functions, $h_1(\alpha)$, $h_2(\beta)$, $k_1(\xi)$, $k_2(\eta)$. However, it is more convenient to express the cross-correlation, $\rho(\tau)$, in terms of the power spectra, $G(\omega)$ and $G_g(\omega)$, and the filter transfer functions, $H_1(\omega)$, $H_2(\omega)$, $K_1(\omega)$, and $K_2(\omega)$. To do this, Eqs. (5.10) and (5.8) are substituted in Eqs. (5.4) and (5.3). In the resulting expression, $R(\tau)$ and $R_g(\tau)$ are replaced by the Fourier integrals which relate them to $G(\omega)$ and $G_g(\omega)$, given by Eq. (A6) of the Appendix. By interchanging these Fourier integrals with the integrals indicated in Eq. (5.5), and performing the latter integrations, we obtain the following expressions for $\rho_n(\tau)$ and $\rho_{mn}(\tau)$, where an asterisk,*, denotes a complex conjugate. In deriving these formulas, we have

made use of the facts that a spectral density is an even function of its argument, while any filter transfer function, say H, is Hermitian; that is, $H(-\omega) = H^*(\omega)$.

$$4\rho_n(\tau) = \tag{5.11}$$
$$K_1(0)\,K_2(0) \int d\omega_1\, G(\omega_1)\, |H_1(\omega_1)|^2 \int d\omega_2\, G(\omega_2)\, |H_2(\omega_2)|^2$$
$$+ 2 \iint d\omega_1\, d\omega_2\, e^{i\tau(\omega_1+\omega_2)}\, G(\omega_1)\, G(\omega_2)\, H_1(\omega_1)\, H_2^*(\omega_1)$$
$$H_1(\omega_2)\, H_2^*(\omega_2)\, K_1(\omega_1+\omega_2)\, K_2^*(\omega_1+\omega_2).$$

$$4\rho_{mn}(\tau) = \tag{5.12}$$
$$K_1(0)\,K_2(0) \Big\{ \int d\omega_1\, G(\omega_1)\, |H_1(\omega_1)|^2 \int d\omega_2\, G(\omega_2)\, |H_2(\omega_2)|^2$$
$$+ \tfrac{1}{2} m^2 \int d\omega_1\, G(\omega_1)\, |H_1(\omega_1)|^2 \int d\nu\, G_g(\nu) \int d\omega_2\, G(\omega_2)\, |H_2(\omega_2+\nu)|^2$$
$$+ \tfrac{1}{2} m^2 \int d\omega_2\, G(\omega_2)\, |H_2(\omega_2)|^2 \int d\nu\, G_g(\nu) \int d\omega_1\, G(\omega_1)\, |H_1(\omega_1+\nu)|^2 \Big\}$$

$$+ 2 \iint d\omega_1\, d\omega_2\, e^{i\tau(\omega_1+\omega_2)}\, G(\omega_1)\, G(\omega_2)\, H_1(\omega_1)\, H_2^*(\omega_1)$$
$$H_1(\omega_2)\, H_2^*(\omega_2)\, K_1(\omega_1+\omega_2)\, K_2^*(\omega_1+\omega_2)$$

$$+ 2m^2 \int d\nu\, G_g(\nu)\, e^{i\nu\tau}\, K_1(\nu)\, K_2^*(\nu)$$
$$\int d\omega_1\, G(\omega_1)\, H_1^*(\omega_1)\, H_1(\omega_1+\nu) \int d\omega_2\, G(\omega_2)\, H_2^*(\omega_2)\, H_2(\omega_2-\nu)$$
$$+ m^2 \int d\nu\, G_g(\nu) \iint d\omega_1\, d\omega_2\, G(\omega_1)\, G(\omega_2)\, e^{i\tau(\omega_1+\omega_2)}$$
$$H_1(\omega_1+\nu)\, H_2^*(\omega_1)\, H_1(\omega_2-\nu)\, H_2^*(\omega_2)\, K_1(\omega_1+\omega_2)\, K_2^*(\omega_1+\omega_2)$$
$$+ m^2 \int d\nu\, G_g(\nu) \iint d\omega_1\, d\omega_2\, G(\omega_1)\, G(\omega_2)\, e^{i\tau(\omega_1+\omega_2)}$$
$$H_1(\omega_1)\, H_2^*(\omega_1-\nu)\, H_1(\omega_2)\, H_2^*(\omega_2+\nu)\, K_1(\omega_1+\omega_2)\, K_2^*(\omega_1+\omega_2)$$
$$+ 2m^2 \int d\nu\, G_g(\nu)\, e^{i\nu\tau} \iint d\omega_1\, d\omega_2\, G(\omega_1)\, G(\omega_2)\, e^{i\tau(\omega_1+\omega_2)}$$
$$H_1(\omega_1+\nu)\, H_2^*(\omega_1+\nu)\, H_1(\omega_2)\, H_2^*(\omega_2)\, K_1(\omega_1+\omega_2+\nu)\, K_2^*(\omega_1+\omega_2+\nu)$$
$$+ 2m^2 \int d\nu\, G_g(\nu)\, e^{i\nu\tau} \iint d\omega_1\, d\omega_2\, G(\omega_1)\, G(\omega_2)\, e^{i\tau(\omega_1+\omega_2)}$$
$$H_1(\omega_1+\nu)\, H_2^*(\omega_1)\, H_1(\omega_2)\, H_2^*(\omega_2+\nu)\, K_1(\omega_1+\omega_2+\nu)\, K_2^*(\omega_1+\omega_2+\nu)$$

The reader should observe that all the foregoing integrals are real valued quantities.

Later, we will assume that $K_1(0) = 0 = K_2(0)$. Therefore, we have written the foregoing expressions for $\rho_n(\tau)$ and $\rho_{mn}(\tau)$ in such a way as to make evident the form they assume in this case.

To conclude this section, we indicate how, with a knowledge of the mean value function Eq. (4.13) and the covariance function Eq. (4.14) of $g(t)$, one is able to evaluate the quantity

$$\lim_{T\to 0} \frac{1}{T} \int_0^T \langle \bar{V}_1(t)\, \bar{V}_2(t) \rangle_{mn} \, dt \tag{5.13}$$

required in Eq. (4.12). Using the methods of this section, it is evident that Eq. (5.13) is equal to

$$\int^1 \langle F_1[y(t)] \rangle \lim_{T\to 0} \frac{1}{T} \int_0^T \langle F_1[1+mg(t)] \rangle . \tag{5.14}$$

Using the expansion of $F_1\left[1+mg(t)\right]$ given by Eq. (5.9), it is easily seen that, up to terms in m^2,

$$\begin{aligned} \lim_{T\to 0} \frac{1}{T} \int_0^T \langle F_1[1+mg(t)] \rangle &= 1 \\ &+ m\left\{ \mu_g(-\alpha_1-\xi) + \mu_g(-\alpha_2-\xi) + \mu_g(\tau-\beta_1-\eta) + \mu_g(\tau-\beta_2-\eta) \right\} \\ &+ m^2 \Big\{ \Gamma_g(-\alpha_1-\xi, -\alpha_2-\xi) + \Gamma_g(\tau-\beta_1-\eta, \tau-\beta_2-\eta) \\ &\quad + \Gamma_g(-\alpha_1-\xi, \tau-\beta_1-\eta) + \Gamma_g(-\alpha_1-\xi, \tau-\beta_2-\eta) \\ &\quad + \Gamma_g(-\alpha_2-\xi, \tau-\beta_1-\eta) + \Gamma_g(-\alpha_2-\xi, \tau-\beta_2-\eta) \Big\} \end{aligned} \tag{5.15}$$

It is clear that if $g(t)$ is a zero mean stationary random time function, then the right hand side of Eq. (5.15) is equal to the right hand side of Eq. (5.10).

Using Eq. (5.15) and Eq. (5.8), one could write an expression for Eq. (5.14) similar to that written for Eq. (5.4) in Eq. (5.12). However, for the present, we leave this computation as it is.

6. AUTOCORRELATION AFTER THE MULTIPLIER

In this section, we compute

$$R_w(\tau) = \left\langle \bar{V}_1(t)\, \bar{V}_2(t)\, \bar{V}_1(t+\tau)\, \bar{V}_2(t+\tau) \right\rangle_n \tag{6.1}$$

the autocorrelation of the output of the multiplier when the input is unmodulated stationary Gaussian noise. Using the same approach as in the previous section, we write

$$R_w(\tau) = \int'' \left\langle F_2 \left[y(t) \right] \right\rangle, \tag{6.2}$$

where for brevity we use the double-primed integral sign $\int''$ to denote the twelve fold integration

$$\begin{aligned} &\iiiint d\xi_1\, d\xi_2\, d\eta_1\, d\eta_2\; k_1(\xi_1)\, k_1(\xi_2)\, k_2(\eta_1)\, k_2(\eta_2) \\ &\iiiint d\alpha_1\, d\alpha_2\, d\alpha_3\, d\alpha_4\; h_1(\alpha_1)\, h_1(\alpha_2)\, h_1(\alpha_3)\, h_1(\alpha_4) \\ &\iiiint d\beta_1\, d\beta_2\, d\beta_3\, d\beta_4\; h_2(\beta_1)\, h_2(\beta_2)\, h_2(\beta_3)\, h_2(\beta_4) \end{aligned} \tag{6.3}$$

and we define the notation $F_2\left[y(t)\right]$ to mean

$$\begin{aligned} &y(t-\alpha_1-\xi_1)\; y(t-\alpha_2-\xi_1)\; y(t+\tau-\alpha_3-\xi_2)\; y(t+\tau-\alpha_4-\xi_2) \\ &y(t-\beta_1-\eta_1)\; y(t-\beta_2-\eta_1)\; y(t+\tau-\beta_3-\eta_2)\; y(t+\tau-\beta_4-\eta_2) \end{aligned} \tag{6.4}$$

Now $\left\langle F_2\left[y(t)\right]\right\rangle$ is an eight fold average of a Gaussian random function. Consequently, by Eq. (A7) of the Appendix, it may be expressed as the sum of 105 terms, each term a product of four

two fold averages. It is easy, although tedious, to enumerate these terms. It is found that, if we assume that

$$K_1(0) = 0 = K_2(0), \tag{6.5}$$

then 45 of these terms vanish when the integrations indicated by Eq. (6.3) are performed. By the same method as used in the previous section, the remaining 60 terms can be expressed in terms of the power spectra $G(\omega)$ and $G_g(\omega)$, and the filter transfer functions $H_1(\omega)$, $H_2(\omega)$, $K_1(\omega)$, and $K_2(\omega)$. We finally obtain the following expression for $R_w(\tau)$.

$$16\, R_w(\tau) = \tag{6.6}$$

$$4\left\{\iint d\omega_1\, d\omega_2\, G(\omega_1) G(\omega_2)\, H_1(\omega_1) H_2^*(\omega_1)\, H_1(\omega_2) H_2^*(\omega_2)\, K_1(\omega_1+\omega_2)\, K_2^*(\omega_1+\omega_2)\right\}^2$$

$$+4 \iiiint d\omega_1\, d\omega_2\, d\omega_3\, d\omega_4\; G(\omega_1) G(\omega_2) G(\omega_3) G(\omega_4)\, e^{i\tau(\omega_1+\omega_2+\omega_3+\omega_4)}$$
$$|H_1(\omega_1)|^2\, |H_2(\omega_2)|^2\, |H_1(\omega_3)|^2\, |H_2(\omega_4)|^2\, |K_1(\omega_1+\omega_3)|^2\, |K_2(\omega_2+\omega_4)|^2$$

$$+4 \iiiint d\omega_1\, d\omega_2\, d\omega_3\, d\omega_4\; G(\omega_1) G(\omega_2) G(\omega_3) G(\omega_4)\, e^{i\tau(\omega_1+\omega_2+\omega_3+\omega_4)}$$
$$H_1(\omega_1) H_2^*(\omega_1)\, H_1(\omega_2) H_2^*(\omega_2)\, H_1^*(\omega_3) H_2(\omega_3)\, H_1^*(\omega_4) H_2(\omega_4)$$
$$K_1(\omega_1+\omega_2)\, K_2^*(\omega_1+\omega_2)\, K_1^*(\omega_3+\omega_4)\, K_2(\omega_3+\omega_4)$$

$$+16 \iiiint d\omega_1\, d\omega_2\, d\omega_3\, d\omega_4\; G(\omega_1) G(\omega_2) G(\omega_3) G(\omega_4)\, e^{i\tau(\omega_1+\omega_2)}$$
$$|H_1(\omega_1)|^2\, |H_2(\omega_2)|^2\, H_1(\omega_3) H_2^*(\omega_3)\, H_1(\omega_4) H_2^*(\omega_4)$$
$$K_1(\omega_1+\omega_3)\, K_2^*(\omega_2+\omega_4)\, K_1^*(\omega_1-\omega_4)\, K_2(\omega_2-\omega_3)$$

$$+16 \iiiint d\omega_1\, d\omega_2\, d\omega_3\, d\omega_4\; G(\omega_1) G(\omega_2) G(\omega_3) G(\omega_4)\, e^{i\tau(\omega_1+\omega_2)}$$
$$H_1(\omega_1) H_2^*(\omega_1)\, H_1^*(\omega_2) H_2(\omega_2)\, H_1(\omega_3) H_2^*(\omega_3)\, H_1(\omega_4) H_2^*(\omega_4)$$
$$K_1(\omega_1+\omega_3)\, K_2^*(\omega_1+\omega_4)\, K_1^*(\omega_2-\omega_4)\, K_2(\omega_2-\omega_3)$$

$$+16 \iiiint d\omega_1\, d\omega_2\, d\omega_3\, d\omega_4\; G(\omega_1) G(\omega_2) G(\omega_3) G(\omega_4)\, e^{i\tau(\omega_1+\omega_2+\omega_3+\omega_4)}$$
$$|H_1(\omega_1)|^2\, |H_2(\omega_2)|^2\, H_1(\omega_3) H_2^*(\omega_3)\, H_1^*(\omega_4) H_2(\omega_4)$$
$$K_1(\omega_1+\omega_3)\, K_2(\omega_2+\omega_4)\, K_1^*(\omega_1+\omega_4)\, K_2^*(\omega_2+\omega_3)$$

7. COMPUTATION OF THE DETECTION CRITERION

In view of Eq. (4.11), the detection criterion D(T) is known for T large as soon as we know

$$\rho_{mn}(0) - \rho_n(0) \tag{7.1}$$

and

$$2\pi \; G'_w(0) \tag{7.2}$$

These quantities may now be expressed in terms of the characteristics of the filters and the statistics of the input noise and modulating function.

From Eqs. (5.11) and (5.12), and the assumption Eq. (6.5), it follows that

$$\rho_{mn}(0) - \rho_n(0) = \tag{7.3}$$

$$\begin{aligned}
&\tfrac{1}{2} m^2 \int d\nu \, G_g(\nu) \; K_1(\nu) \; K_2^*(\nu) \int d\omega_1 \, G(\omega_1) \; H_1^*(\omega_1) H_1(\omega_1+\nu) \\
&\qquad \int d\omega_2 \; G(\omega_2) \; H_2^*(\omega_2) \; H_2(\omega_2+\nu) \\
&+\tfrac{1}{4} m^2 \int d\nu \; G_g(\nu) \iint d\omega_1 \, d\omega_2 \; G(\omega_1) \; G(\omega_2) \; H_1(\omega_1+\nu) \; H_2^*(\omega_1) \\
&\qquad H_1(\omega_1+\nu) \;\; H_2^*(\omega_2) \;\; K_1(\omega_1+\omega_2) \;\; K_2^*(\omega_1+\omega_2) \\
&+\tfrac{1}{4} m^2 \int d\nu \; G_g(\nu) \iint d\omega_1 \, d\omega_2 \; G(\omega_1) \; G(\omega_2) \; H_1(\omega_1) \; H_2^*(\omega_1-\nu) \\
&\qquad H_1(\omega_2) \;\; H_2^*(\omega_2+\nu) \;\; K_1(\omega_1+\omega_2) \;\; K_2^*(\omega_1+\omega_2) \\
&+\tfrac{1}{2} m^2 \int d\nu \; G_g(\nu) \iint d\omega_1 \, d\omega_2 \; G(\omega_1) \; G(\omega_2) \; H_1(\omega_1+\nu) \; H_2^*(\omega_1+\nu) \\
&\qquad H_1(\omega_2) \;\; H_2^*(\omega_2) \;\; K_1(\omega_1+\omega_2+\nu) \;\; K_2^*(\omega_1+\omega_2+\nu) \\
&+\tfrac{1}{2} m^2 \int d\nu \; G_g(\nu) \iint d\omega_1 \, d\omega_2 \; G(\omega_1) \; G(\omega_2) \; H_1(\omega_1+\nu) \; H_2^*(\omega_1) \\
&\qquad H_1(\omega_2) \;\; H_2^*(\omega_2+\nu) \;\; K_1(\omega_1+\omega_2+\nu) \;\; K_2^*(\omega_1+\omega_2+\nu) \,.
\end{aligned}$$

It is immediately seen from Eq. (5.11) that, under the assumption Eq. (6.5), the first term in the expression Eq. (6.6) for $R_w(\tau)$ is equal to $\rho_n^2(0)$. Therefore $R'_w(\tau)$ is given by the other terms of Eq. (6.6). By integrating these terms with respect to τ from $-\infty$ to ∞, we obtain the following expression.

$$
\begin{aligned}
2\pi G'_w(0) = 4\pi \Big\{ & \frac{1}{4} \int d\mu \, |K_1(\mu)|^2 \, |K_2(\mu)|^2 \int d\omega_1 \, G(\omega_1) \, G(\omega_1+\mu) \, |H_1(\omega_1)|^2 \, |H_1(\omega_1+\mu)|^2 \\
& \quad \int d\omega_2 \, G(\omega_2) \, G(\omega_2+\mu) \, |H_2(\omega_2)|^2 \, |H_2(\omega_2+\mu)|^2 \\
& + \frac{1}{4} \int d\mu \, \{K_1^*(\mu) \, K_2(\mu)\}^2 \, \Big\{ \int d\omega \, G(\omega) \, G(\omega+\mu) \, H_1(\omega) \, H_2^*(\omega) \\
& \quad H_1^*(\omega+\mu) \, H_2(\omega+\mu) \Big\}^2 \\
& + \int d\mu \, |G(\mu)|^2 \, |H_1(\mu)|^2 \, |H_2(\mu)|^2 \\
& \quad \Big| \int d\omega \, K_1(\omega) \, K_2^*(\omega) \, G(\omega+\mu) \, H_1(\omega+\mu) \, H_2^*(\omega+\mu) \Big|^2 \\
& + \int d\mu \, |G(\mu)|^2 \, \{H_1^*(\mu) \, H_2(\mu)\}^2 \\
& \quad \Big\{ \int d\omega \, K_1(\omega) \, K_2^*(\omega) \, G(\omega+\mu) \, H_1(\omega+\mu) \, H_2^*(\omega+\mu) \Big\}^2 \\
& + \int d\mu \, K_1^*(\mu) \, K_2(\mu) \iint d\omega_1 \, d\omega_2 \, G(\omega_1) \, G(\omega_1+\mu) \, G(\omega_2) \, G(\omega_2+\mu) \\
& \quad |H_1(\omega_1)|^2 \, H_1(\omega_1+\mu) \, H_2^*(\omega_1+\mu) \, |H_2(\omega_2)|^2 \, H_1(\omega_2+\mu) \, H_2^*(\omega_2+\mu) \\
& \quad K_1(\omega_1+\omega_2+\mu) \, K_2^*(\omega_1+\omega_2+\mu) \Big\}
\end{aligned}
\tag{7.4}
$$

From the above results, we may immediately obtain the detection criterion D(T) for the system of Fig. 2 by setting $H_1(\omega) = H_2(\omega) = H(\omega)$ and $K_1(\omega) = K_2(\omega) = K(\omega)$. D(T) for T large is still given by Eq. (4.11), but now the terms on the right hand side of Eq. (4.11) may be expressed as follows:

$$P_{mn}(0) - P_n(0) = \tag{7.5}$$

$$\frac{1}{2} m^2 \int d\nu\, G_g(\nu)\, |K(\nu)|^2 \left| \int d\omega\, G(\omega)\, H^*(\omega)\, H(\omega+\nu) \right|^2$$

$$+ \frac{1}{2} m^2 \int d\nu\, G_g(\nu) \iint d\omega_1\, d\omega_2\, G(\omega_1)\, G(\omega_2)\, H^*(\omega_1)\, H(\omega_1+\nu)$$
$$H^*(\omega_2)\, H(\omega_2-\nu)\, |K(\omega_1+\omega_2)|^2$$

$$+ \frac{1}{2} m^2 \int d\nu\, G_g(\nu) \iint d\omega_1\, d\omega_2\, G(\omega_1)\, G(\omega_2)\, |H(\omega_1)|^2$$
$$|H(\omega_2+\nu)|^2\, |K(\omega_1+\omega_2+\nu)|^2$$

$$+ \frac{1}{2} m^2 \int d\nu\, G_g(\nu) \iint d\omega_1\, d\omega_2\, G(\omega_1)\, G(\omega_2)\, H^*(\omega_1)\, H(\omega_1+\nu)$$
$$H(\omega_2)\, H^*(\omega_2+\nu)\, |K(\omega_1+\omega_2+\nu)|^2$$

$$2\pi\, G'_{w}(0) = 4\pi \tag{7.6}$$

$$\Big\{ \frac{1}{2} \int d\mu\, |K(\mu)|^4 \left\{ \int d\omega\, G(\omega)\, G(\omega+\mu)\, |H(\omega)|^2\, |H(\omega+\mu)|^2 \right\}^2$$

$$+ 2 \int d\mu\, |G(\mu)|^2\, |H(\mu)|^4 \left\{ \int d\omega\, |K(\omega)|^2\, G(\omega+\mu)\, |H(\omega+\mu)|^2 \right\}^2$$

$$+ \int d\mu\, |K(\mu)|^2 \iint d\omega_1\, d\omega_2\, G(\omega_1)\, G(\omega_2)\, G(\omega_1+\mu)\, G(\omega_2+\mu)$$
$$|H_1(\omega_1)|^2\, |H(\omega_1+\mu)|^2\, |H(\omega_2)|^2\, |H(\omega_2+\mu)|^2\, |K(\omega_1+\omega_2+\mu)|^2 \Big\}$$

8. APPLICATION TO THE CASE OF GAUSSIAN FILTERS

To illustrate these results, let us compute D(T) for the system of Fig. 2. For mathematical convenience, let us assume that, up to phase factors, the filter transfer functions $H(\omega)$ and $K(\omega)$ are given by Gaussian functions, as follows:

$$H(\omega) = \exp\left[-\frac{1}{2}\left(\frac{\omega-\Omega_H}{\sigma_H}\right)^2\right] + \exp\left[-\frac{1}{2}\left(\frac{\omega+\Omega_H}{\sigma_H}\right)^2\right] \tag{8.1}$$

$$K(\omega) = \exp\left[-\frac{1}{2}\left(\frac{\omega-\Omega_K}{\sigma_K}\right)^2\right] + \exp\left[-\frac{1}{2}\left(\frac{\omega+\Omega_K}{\sigma_K}\right)^2\right] \tag{8.2}$$

While not physically realizable, Gaussian shapped filters are often good approximations, for mathematical purposes, of actual filters. Under this assumption, the evaluation of the integrals in Eqs. (7.5) and (7.6) is much simpler than it would be otherwise, for we may use the following useful formula for the product of two Gaussian factors:

$$\begin{aligned} &\exp\left[-\frac{1}{2}\left(\frac{\omega-\Omega_1}{\sigma_1}\right)^2\right] \exp\left[-\frac{1}{2}\left(\frac{\omega-\Omega_2}{\sigma_2}\right)^2\right] \\ &= \exp\left[-\frac{1}{2}\left(\frac{\omega-\Omega}{\sigma}\right)^2\right] \exp\left[-\frac{1}{2}\frac{(\Omega_1-\Omega_2)^2}{\sigma_1^2+\sigma_2^2}\right] \end{aligned} \tag{8.3}$$

where

$$\Omega = \frac{\Omega_1\sigma_2^2 + \Omega_2\sigma_1^2}{\sigma_1^2+\sigma_2^2}, \quad \sigma^2 = \frac{\sigma_1^2\sigma_2^2}{\sigma_1^2+\sigma_2^2} \tag{8.4}$$

It is easy to verify Eq. (8.3) by expanding both sides. From Eq. (8.3) it follows that

$$\int d\omega \, \exp\left[-\tfrac{1}{2}\left(\frac{\omega-\Omega_1}{\sigma_1}\right)^2\right] \exp\left[-\tfrac{1}{2}\left(\frac{\omega-\Omega_2}{\sigma_2}\right)^2\right] \tag{8.5}$$
$$= \sqrt{2\pi}\left[\frac{\sigma_1^2\sigma_2^2}{\sigma_1^2+\sigma_2^2}\right]^{1/2} \exp\left[-\frac{1}{2}\,\frac{(\Omega_1-\Omega_2)^2}{\sigma_1^2+\sigma_2^2}\right]$$

For the signal power spectrum $G_g(\omega)$, we will assume a Gaussian function centered at roughly the same point as is $K(\omega)$.

$$G_g(\omega) = P_g \frac{1}{\sqrt{2\pi}}\, \sigma_g^{-1}\left\{\exp\left[-\tfrac{1}{2}\left(\frac{\omega-\Omega_K}{\sigma_g}\right)^2\right] + \exp\left[-\tfrac{1}{2}\left(\frac{\omega+\Omega_K}{\sigma_g}\right)^2\right]\right\} \tag{8.6}$$

There is no difficulty, however, in treating any other form of signal spectrum.

The noise power spectrum $G(\omega)$ is assumed to be flat and identically equal to a constant G.

We assume that all cross-product terms may be ignored when these Gaussian functions are multiplied. We then have by Eq. (8.3) that

$$H(\omega)\,H(\omega+\mu) = \exp\left[-\tfrac{1}{2}\,\frac{\mu^2}{2\sigma_H^2}\right] \tag{8.7}$$
$$\left\{\exp\left[-\tfrac{1}{2}\,\frac{(\omega-\Omega_H+\mu/2)^2}{\sigma_H^2/2}\right] + \exp\left[-\tfrac{1}{2}\,\frac{(\omega+\Omega_H+\mu/2)^2}{\sigma_H^2/2}\right]\right\}$$

We also sssume that

$$\frac{\sigma_H^{\,2}}{\sigma_H^{\,2}+\sigma_K^{\,2}+\sigma_g^{\,2}} \simeq 1\,, \quad \exp\left[-\left(\frac{\Omega_K}{\sigma_H}\right)^2\right] \doteq 1\,. \tag{8.8}$$

There is now no difficulty in evaluating Eqs. (7.5) and (7.6). We obtain the following approximate expressions:

$$P_{mn}(0) - P_n(0) = \tag{8.9}$$

$$4\pi\, m^2 P_g\, G^2 \sigma_H^{\,2} \left\{ \left[1+\left(\frac{\sigma_g}{\sigma_K}\sqrt{2}\right)^2\right]^{-1/2} + \frac{3}{\sqrt{2}}\,\frac{\sigma_K}{\sigma_H} \right\}$$

$$2\pi\, G'_w(0) =$$

$$2\pi\left(\sqrt{2\pi}\right)^3 G^4\, \sigma_H^{\,2}\, \sigma_K \left\{1+\left(4\sqrt{2}+2\right)\frac{\sigma_K}{\sigma_H}\right\} \tag{8.10}$$

As a measure of the ratio of the bandwidth of the spectrum of the modulating function to the bandwidth of the power transfer functions $|H(\omega)|^2$ and $|K(\omega)|^2$ of the filters H and K, define, respectively,

$$R_H = \frac{\sigma_g}{\sigma_H}\sqrt{2}\,, \qquad R_K = \frac{\sigma_g}{\sigma_K}\sqrt{2}\,. \tag{8.11}$$

Then, by means of Eq. (4.11), we have approximately,

$$\frac{D^2(T)}{T} = \frac{1}{\sqrt{\pi}} \left[2m^2 P_g \right]^2 \frac{\sigma_H^{\,2}}{\sigma_g} \frac{\left\{ R_K \left(1 + R_K^{\,2}\right)^{-1/2} + \frac{3}{\sqrt{2}} R_H \right\}^2}{R_K + \left(4\sqrt{2} + 2\right) R_H} \tag{8.12}$$

9. EXTENSION TO THE INCLUSION OF BACKGROUND NOISE

It is of some interest to consider the problem of detecting the amplitude modulated noise in the presence of background noise.

We let $z(t)$ denote a stationary Gaussian random time function, with autocorrelation $R_z(\tau)$ and power spectrum $G_z(\omega)$. We assume $z(t)$ to be statistically independent of $y(t)$ and $g(t)$. Let

$$y_1(t) = y(t) + z(t). \tag{9.1}$$

Then $y_1(t)$ is a stationary Gaussian random time function with autocorrelation

$$R_1(\tau) = R_y(\tau) + R_z(\tau) \tag{9.2}$$

and power spectrum

$$G_1(\omega) = G_y(\omega) + G_z(\omega). \tag{9.3}$$

If the input of the system of Fig. 1 is $y_1(t)$, then the cross-correlation of the outputs before the multiplier (denoted by $\rho_{n+b}(\tau)$) and the autocorrelation of the output after the multiplier, continue to be given by Eqs. (5.11) and (6.6), respectively, with the proviso that instead of $G(\omega)$ we read $G_1(\omega)$.

Next, let us consider the case where the input to the system of Fig. 1 is

$$u(t) = y(t)\left[1 + m\, g(t)\right] + z(t), \tag{9.4}$$

and let us compute the cross-correlation of the outputs before the multiplier, which we denote by $\rho_{mn+b}(\tau)$. In the notation of Sec. 5,

$$\rho_{mn+b}(\tau) = \int' \left\langle F_1\left[u(t)\right]\right\rangle_{TAv} \tag{9.5}$$

Using the methods of Sec. 5, it may be shown that

$$4\rho_{mn+b}(\tau) = 4\rho_{n+b}(\tau) \tag{9.6}$$

$$+ K_1(0)\,K_2(0)\,\frac{m^2}{2}\Big\{\int d\omega_1\, G_y(\omega_1)\,|H_1(\omega_1)|^2 \int d\nu\, G_y(\nu) \int d\omega_2\, G_y(\omega_2)\,|H_2(\omega_2+\nu)|^2$$

$$+ \int d\omega_2\, G_y(\omega_2)\,|H_2(\omega_2)|^2 \int d\nu\, G_y(\nu) \int d\omega_1\, G_y(\omega_1)\,|H_1(\omega_1+\nu)|^2$$

$$+ \int d\omega_1\, G_z(\omega_1)\,|H_1(\omega_1)|^2 \int d\nu\, G_z(\nu) \int d\omega_2\, G_y(\omega_2)\,|H_2(\omega_2+\nu)|^2$$

$$+ \int d\omega_2\, G_z(\omega_2)\,|H_2(\omega_2)|^2 \int d\nu\, G_z(\nu) \int d\omega_1\, G_y(\omega_1)\,|H_1(\omega_1+\nu)|^2\Big\}$$

$$+ 2m^2 \int d\nu\, G_g(\nu)\, e^{i\nu\tau}\, K_1(\nu)\, K_2^*(\nu) \int d\omega_1\, G_y(\omega_1)\, H_1^*(\omega_1)\, H_1(\omega_1+\nu)$$
$$\int d\omega_2\, G_y(\omega_2)\, H_2^*(\omega_2)\, H_2(\omega_2-\nu)$$

$$+ m^2 \int d\nu\, G_g(\nu) \iint d\omega_1\, d\omega_2\, G_y(\omega_1)\, G_y(\omega_2)\, e^{i\tau(\omega_1+\omega_2)}\, H_1(\omega_1)\, H_2^*(\omega_1+\nu)$$
$$H_1(\omega_2)\, H_2^*(\omega_2-\nu)\quad K_1(\omega_1+\omega_2)\, K_2^*(\omega_1+\omega_2)$$

$$+ m^2 \int d\nu\, G_g(\nu) \iint d\omega_1\, d\omega_2\, G_y(\omega_1)\, G_y(\omega_2)\, e^{i\tau(\omega_1+\omega_2)}\, H_1(\omega_1+\nu)\, H_2^*(\omega_2)$$
$$H_1(\omega_1-\nu)\, H_2^*(\omega_2)\quad K_1(\omega_1+\omega_2)\, K_2^*(\omega_1+\omega_2)$$

$$+ 2m^2 \int d\nu\, G_z(\nu)\, e^{i\nu\tau} \iint d\omega_1\, d\omega_2\, G_y(\omega_1)\, G_y(\omega_2)\, e^{i\tau(\omega_1+\omega_2)}\, H_1(\omega_1)\, H_2^*(\omega_1)$$
$$H_1(\omega_2+\nu)\, H_2^*(\omega_2+\nu)\quad K_1(\omega_1+\omega_2+\nu)\quad K_2^*(\omega_1+\omega_2+\nu)$$

$$+ 2m^2 \int d\nu\, G_y(\nu)\, e^{i\nu\tau} \iint d\omega_1\, d\omega_2\, G_y(\omega_1)\, G_y(\omega_2)\, e^{i\tau(\omega_1+\omega_2)}\, H_1(\omega_1+\nu)\, H_2^*(\omega_1)$$
$$H_1(\omega_2)\, H_2^*(\omega_2+\nu)\quad K_1(\omega_1+\omega_2+\nu)\quad K_2^*(\omega_1+\omega_2+\nu)$$

$$+ 2m^2 \int d\nu\, G_y(\nu)\, e^{i\nu\tau} \iint d\omega_1\, d\omega_2\, G_z(\omega_1)\, G_y(\omega_2)\, e^{i\tau(\omega_1+\omega_2)}\, H_1(\omega_1)\, H_2^*(\omega_1)$$
$$H_1(\omega_2+\nu)\, H_2^*(\omega_2+\nu)\quad K_1(\omega_1+\omega_2+\nu)\quad K_2^*(\omega_1+\omega_2+\nu)$$

The reader should observe that the difference

$$\left[\rho_{mn+b}(\tau) - \rho_{n+b}(\tau)\right] - \left[\rho_{mn}(\tau) - \rho_n(\tau)\right] \tag{9.7}$$

is given, up to a constant, by the very last term of Eq. (9.6), under the assumption of Eq. (6.5). Thus the average mean level out of the system due to the presence of the modulating function is increased by this term when background noise is present. However, the fluctuation term in the denominator of the detection criterion is greatly increased by the background noise. Consequently, as one naturally expects, the detectability (as given by the detection criterion) decreases as the background noise increases.

10. CONCLUSIONS

In the foregoing, we have developed formulae which enable us to study the behavior, and the optimum design, of the detection system of Fig. 1. We have made three calculations which may be of general interest. We have (1) introduced, and derived various limiting forms for the detection criterion; (2) computed the cross-correlation of the outputs entering the multiplier for two kinds of input, stationary Gaussian noise and amplitude modulated noise; and (3) computed the autocorrelation of the output of the multiplier when the input is noise. From the correlation function, the corresponding power spectrum can be obtained by means of the Wiener-Khintchine relations.

The mathematical techniques used here may be of use in many other contexts than the one we have explicitly considered. In the calculation of (2) and (3), a basic role was played by Eq. (A7) of the Appendix, which gives an explicit expansion of the higher order statistics of a Gaussian random process in terms of its second order statistics. By using this expansion, we were able to avoid using the Fourier representation of a Gaussian random process used by many authors[1, 2, 3]. It appears to us that, as long as only linear and quadratic devices are considered, the use of the Fourier representation renders many computations unnecessarily cumbersome, and may not always readily yield the correct result in complicated situations in which delta functions are involved in the power spectra. It has often been observed that the correlation function is generally better behaved than the power spectrum and consequently the mathematical analysis may sometimes be simpler if one first computes the correlation function, instead of the power spectra. The computation of the correlation function can in turn be facilitated by use of formula (A7).

APPENDIX: THE MATHEMATICS OF NOISE

In this appendix, we summarize the main notions in regard to random time functions, and define what is mathematically meant by noise.

Let $y(t)$ be a function of time t defined, for the sake of generality, for $-\infty < t < \infty$. If $y(t)$ contains, in addition to t, random parameters, so that at each point t, $y(t)$ does not have a definite value but only an ensemble of possible values which it assumes in accordance with some given probability distribution, then $y(t)$ is said to be a random time function. Then the ensemble average, denoted by $\langle \cdots \rangle$, is an average taken with respect to these random parameters. We let

$$\mu_y(t) = \langle y(t) \rangle \tag{A1}$$

$$\Gamma_y(t_1, t_2) = \langle y(t_1)\, y(t_2) \rangle \tag{A2}$$

denote the one- and two-point ensemble averages of $y(t)$; $\mu_y(t)$ is called the mean-value function, and $\Gamma_y(t_1, t_2)$ is called the covariance function of $y(t)$.

A random time function for which the ensemble average $\langle y(t)\, y(t+\tau) \rangle$ has a value that is independent of t, and depends only on τ, is called <u>stationary</u>.

The <u>autocorrelation</u> $R_y(\tau)$ of a random time function is defined (for $-\infty < \tau < \infty$) by

$$R_y(\tau) = \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} \langle y(t)\, y(t+\tau) \rangle \, dt \tag{A3}$$
$$= \left[\Gamma_y(t, t+\tau) \right]_{TAv}$$

where the subscript TAv is used to denote a time average, defined as in Eq. A3.

Clearly, for a stationary function

$$R_y(\tau) = \Gamma_y(t, t+\tau) \tag{A4}$$

The <u>power spectrum</u> density function $G_y(\omega)$ is related to $R_y(\tau)$ by means of a Fourier transformation (Wiener-Khintchine theorem)[3].

$$G_y(\omega) = \frac{1}{\pi}\int_{-\infty}^{\infty} e^{-i\tau\omega}\, R_y(\tau)\, d\tau \tag{A5}$$
$$= \frac{2}{\pi}\int_{0}^{\infty} \cos\tau\omega\, R_y(\tau)\, d\tau$$

$$R_y(\tau) = \frac{1}{2}\int_{-\infty}^{\infty} e^{i\tau\omega}\, G_y(\omega)\, d\omega \tag{A6}$$
$$= \int_{0}^{\infty} \cos\tau\omega\, G_y(\omega)\, d\omega$$

By <u>noise</u> is generally meant a stationary random time function, $y(t)$, whose distribution is Gaussian. The important mathematical property of such a random function is that its complete probability distribution, and all possible moments, are fully determined once we know the mean value function, $\mu_y(t)$, and the autocorrelation function, $R_y(\tau)$. The mean value function, $\mu_y(t)$, is often assumed to be identically zero. For such noise, all odd order moments vanish, and the even order moments may be expressed in terms of the second order moments by means of the following useful formula. Let n be an even integer, and let $t_1, \ldots\ldots, t_n$ be points of time, some of which may coincide. Then

$$\langle y(t_1)\cdots y(t_n)\rangle = \sum \langle y(t_{i_1})\, y(t_{i_2})\rangle \cdots \langle y(t_{i_{n-1}})\, y(t_{i_n})\rangle \tag{A7}$$

where the sum is taken over all possible ways of dividing the n points into $n/2$ combinations of 2 pairs. The number of terms in the summa-

tion is equal to $1 \cdot 3 \cdots (n-3)(n-1)$. Thus, for $n = 4$ there are three terms in the sum, and for $n = 8$ there are 105 terms.

It is interesting to observe that Eq. (A7) characterizes noise. A zero mean stationary random time function is Gaussian if, and only if, all its odd moments are zero, and its even moments satisfy Eq. (A7)[4].

By random noise is generally meant noise with a power spectrum that is constant up to quite large frequencies. The covariance function of random noise is thus, more or less, a δ function.

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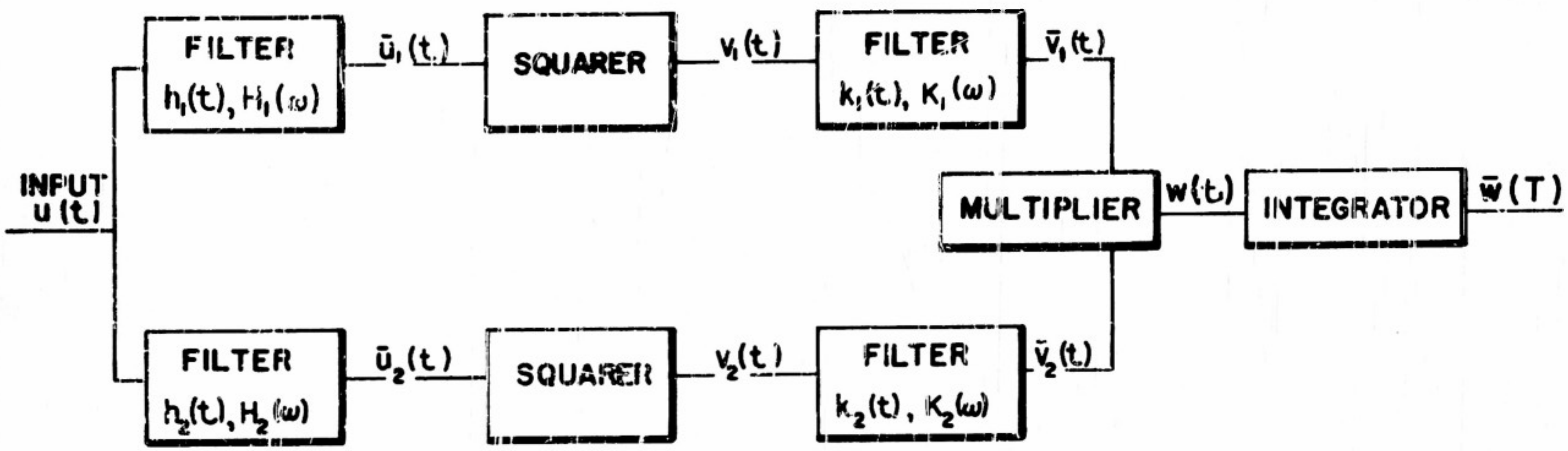

$$\bar{u}_i(t) = \int h_i(\alpha)\, u(t-\alpha)\, d\alpha \qquad i = 1, 2$$
$$v_i(t) = [\bar{u}_i(t)]^2 \qquad i = 1, 2$$
$$\bar{v}_i(t) = \int k_i(\xi)\, v(t-\xi)\, d\xi \qquad i = 1, 2$$
$$w(t) = \bar{v}_1(t)\, \bar{v}_2(t)$$
$$\bar{w}(T) = \int_0^T w(t)\, dt$$

FIRST FILTERS HAVE IMPULSE TRANSFER FUNCTIONS $h_i(t)$, FREQUENCY RESPONSE FUNCTIONS $H_i(\omega)$;

SECOND FILTERS HAVE IMPULSE TRANSFER FUNCTIONS $k_i(t)$, FREQUENCY RESPONSE FUNCTIONS $K_i(\omega), K_i(0) = 0$

FIG. I

GENERAL DETECTION SYSTEM

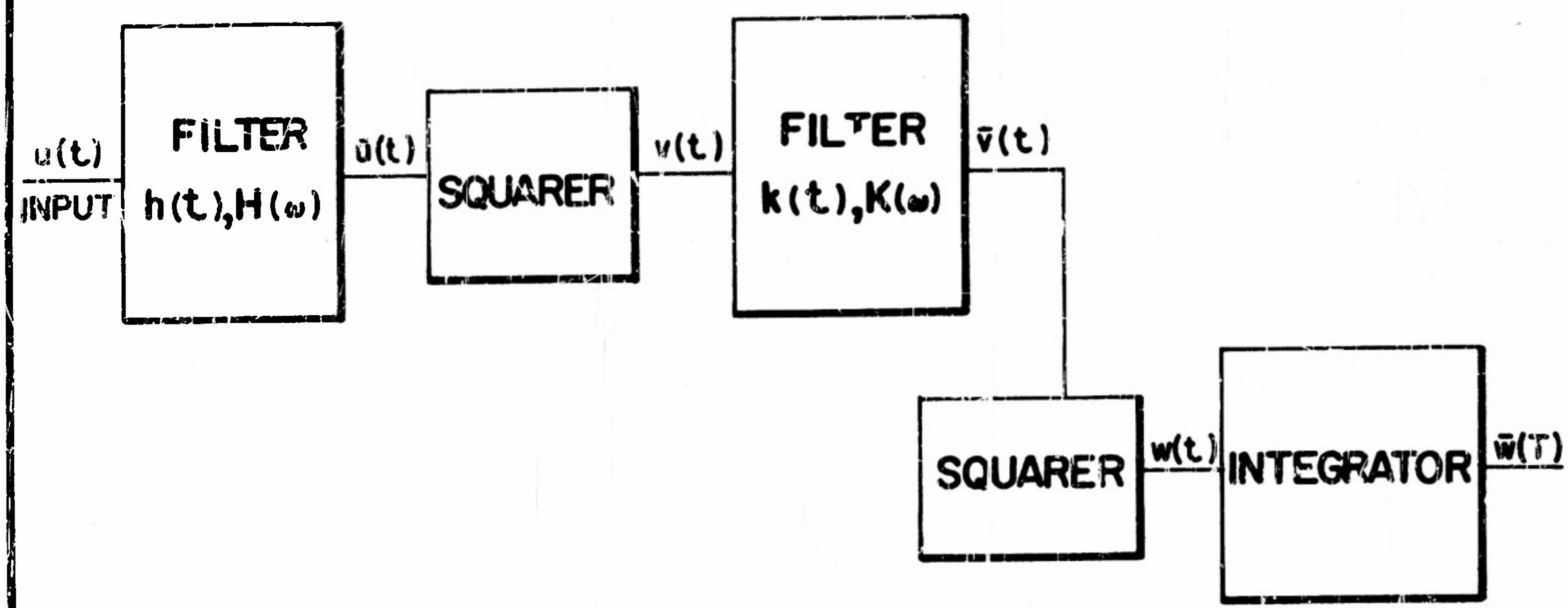

FIG. 2

A TYPICAL DETECTION SCHEME FOR AMPLITUDE MODULATED NOISE

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